Exact coherent states of a noninteracting Fermi gas in a harmonic trap

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Abstract

Exact and closed-form expressions of the particle density, the kinetic energy density, the probability current density, and the momentum distribution are derived for a coherent state of a noninteracting Fermi gas, while such a state can be obtained from the ground state in a d-dimensional isotropic harmonic trap by modulating the trap frequency and shifting the trap center. Conservation laws for the relations of the densities are also given. The profile of the momentum distribution turns out to be identical in shape with that of the particle density, however, the dispersion of the distribution increases (decreases) when that of the particle density is decreased (increased). The expressions are also applicable for a sudden and total opening of the trap, and it is shown that, after the opening, the gas has a stationary momentum distribution whose dispersion could be arbitrarily large or small.

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Experiments on quantum degenerate Fermi atoms provide motivation for the theoretical analysis of the noninteracting Fermi gas in a harmonic trap. Since the interaction energy per particle is small compared with the Fermi energy, the interactions between atoms are controllable, and since the short-range interaction may be ignored in the long-term behavior of the expanding Fermi gas, theoretical studies of noninteracting particles are important for understanding experiments on degenerate Fermi gases [1]. Indeed, in the deep BCS limit, the Fermi gases of atoms are considered to be noninteracting [2, 3]. As most experiments are performed in harmonic traps, analytic expressions of the particle density and the kinetic energy density have been given for the $(M+1)-$shell filled ground state of the Fermi gas in a d-dimensional isotropic harmonic trap [4].

For an isotropic harmonic oscillator, the time-dependent Schrödinger equation is invariant under the displacement-type transformation (DTT), and under squeeze-type transformation (STT) with a rescaling of time. As implied by Kohn's theorem, the system of a single-component gas is still invariant under the extended DTT, which can be used to find a coherent state of the harmonic motion of the center of mass [5, 6]. The system of a noninteracting gas is invariant under the extended STT, which predicts a generalized coherent (squeezed) state of breathing (compressional) motion [6, 7, 8]. In experiments, the system of the (generalized) *coherent state* can be prepared by exciting the system in an eigenstate through shifting the trap center and modulating the trap frequency. For the one-dimensional boson gas of infinitely strong repulsive interaction of zero-range (Tonks-Girardeau gas), the particle density is identical to that of a noninteracting Fermi gas, and the invariance under the STT may be applicable to some extent [6, 8]. Recently invariance under the STT has been used to analytically explain the "dynamical fermionization" of the momentum distribution of the expanding Tonks-Girardeau gas [9, 10].

In this paper, exact and closed-form expressions for the particle density, the kinetic energy density, the probability current density, and the momentum distribution are presented for a state *coherently excited* from the $(M+1)-$shell filled ground state of a noninteracting Fermi gas in a d-dimensional isotropic harmonic trap, and conservation laws for the relations of the densities are given. The particle density and the momentum distribution, which have been studied as a function of the scattering length through a mean-field theory in Ref. [3], are of great experimental interest [2, 9], and the expressions presented here are also valid for the analysis of a sudden and total opening of the trap. The profile of the

momentum distribution turns out to be identical in shape with that of the particle density, however, as a manifestation of the Heisenberg *uncertainty principle*, the dispersion of the distribution increases (decreases) when that of the particle density is decreased (increased). The dispersion of the distribution could be *arbitrarily* large or small, as the amplitude of the breathing motion could be [6]. Though only the coherent states excited from a ground state will be considered here, since the large or small dispersion basically comes from the invariance under the STT and the uncertainty principle, *general* coherent states could *also* have the large or small dispersion. After a total opening of the trap, we show that the momentum distribution of the free gas remains stationary without changing from that of the gas at the moment of the opening. Since we ignore the interactions, we consider only the spin-polarized case.

For a d-dimensional isotropic harmonic oscillator described by the Hamiltonian

$$H(\mathbf{r},\mathbf{p},t) = \sum_{i=1}^{d}\left(\frac{{p_i}^2}{2m} + \frac{1}{2}mw^2(t){x_i}^2 - F_i(t)x_i\right) \tag{1}$$

with m, $w(t)$, and $F_i(t)$ denoting the positive mass, real frequency, and an external force in the i-th direction, respectively, through the extension of the one-dimension result [11], a complete set of the wave functions $\psi_{\mathbf{n}}(\mathbf{r},t)$ satisfying the time-dependent Schrödinger equation is found, with $\mathbf{n} = (n_1, n_2, \ldots, n_d)$, as

$$\psi_{\mathbf{n}}(\mathbf{r},t) = \prod_{i=1}^{d}\psi_{n_i}(x_i, x_i^p, t), \tag{2}$$

where n_i is a non-negative integer, and

$$\begin{aligned}&\psi_{n_i}(x_i, x_i^p, t)\\ &= \frac{1}{\sqrt{2^{n_i}n_i!}\pi^{1/4}\sqrt{l}}\left[\frac{u-iv}{\eta(t)}\right]^{n_i+\frac{1}{2}}\exp\left[\frac{i}{\hbar}\left(\delta_i + m\dot{x}_i^p x_i\right)\right]\\ &\quad\times\exp\left[\frac{(x_i-x_i^p)^2}{2}\left(-\frac{1}{l^2}+i\frac{m\dot{l}}{\hbar l}\right)\right]H_{n_i}\left(\frac{x_i-x_i^p}{l}\right),\end{aligned} \tag{3}$$

with u, v being two linearly independent real homogeneous solutions of the classical equation of motion

$$\ddot{x}_{cl} + w^2(t)x_{cl} = \frac{F_i(t)}{m}, \tag{4}$$

and $\eta(t)$, l being defined as

$$\eta(t) = \sqrt{u^2+v^2}, \qquad l = \eta\sqrt{\frac{\hbar}{\Omega}}, \tag{5}$$

respectively, while a positive constant Ω is defined as $\Omega = m[\dot{v}u - \dot{u}v]$, and the overdots denote differentiations with respect to t. In Eq. (3), H_{n_i} is the Hermite polynomial, x_i^p denotes a solution of Eq. (4), and δ_i is defined through $\dot{\delta}_i = \frac{m}{2}\left[w^2(t)x_i^p - \dot{x}_i^p\right]$.

When $w(t)$ is a constant w_c and $F_i(t) = 0$, if we choose $u = \cos(w_c t)$, $v = \sin(w_c t)$, and $x_i^p = 0$, then $l = \sqrt{\hbar/(m w_c)}$, and $\psi_{\mathbf{n}}(\mathbf{r}, t)$ is the wave function of an eigenstate. The $(M+1)-$shell filled coherent state can be obtained from the $(M+1)-$shell filled ground state, by applying an external force $F_i(t)$ and modulating $w(t)$. If the coherent state is established once, the system remains a coherent state in general even after the external force is turned off and the modulation is stopped [6]. The one-particle reduced density matrix of the $(M+1)-$shell filled coherent state is given by

$$\rho(\mathbf{r}', \mathbf{r}; t) = \sum_{\{\mathbf{n} | \sum_i n_i \leq M\}} \psi^*_{\mathbf{n}}(\mathbf{r}', t)\psi_{\mathbf{n}}(\mathbf{r}, t). \tag{6}$$

With an independent variable b, we define

$$C(\mathbf{r}', \mathbf{r}; t; b) = \sum_{\{\mathbf{n}\}} e^{-b\left(d/2 + \sum_{i=1}^d n_i\right)} \psi^*_{\mathbf{n}}(\mathbf{r}', t)\psi_{\mathbf{n}}(\mathbf{r}, t). \tag{7}$$

Making use of this fact

$$\begin{aligned}
&\exp\left[-\frac{1}{2}\left(\tilde{x}^2 + \tilde{y}^2\right)\right] \sum_{n=0} \frac{e^{-b(1/2+n)}}{2^n n!} H_n(\tilde{x}) H_n(\tilde{y}) \\
&= \frac{\exp\left[-\frac{(\tilde{x}+\tilde{y})^2}{4}\tanh\left(\frac{b}{2}\right) - \frac{(\tilde{x}-\tilde{y})^2}{4}\coth\left(\frac{b}{2}\right)\right]}{\sqrt{2\sinh(b)}}
\end{aligned} \tag{8}$$

which comes from Mehler's formula [12], we find that

$$\begin{aligned}
&C(\mathbf{r}', \mathbf{r}; t; b) \\
&= \frac{1}{(2\pi)^{d/2} l^d \sinh^{d/2}(b)} \\
&\quad \times \exp\left[i\frac{m}{\hbar}\left(\dot{\mathbf{r}}_p \cdot \mathbf{s} + \frac{\dot{l}}{l}\mathbf{s}\cdot(\mathbf{q} - \mathbf{r}_p)\right)\right] \\
&\quad \times \exp\left[-\frac{(\mathbf{q} - \mathbf{r}_p)^2}{l^2}\tanh\left(\frac{b}{2}\right) - \frac{\mathbf{s}^2}{4l^2}\coth\left(\frac{b}{2}\right)\right],
\end{aligned} \tag{9}$$

where

$$\mathbf{q} = (\mathbf{r} + \mathbf{r}')/2, \quad \mathbf{s} = \mathbf{r} - \mathbf{r}', \tag{10}$$

with $\mathbf{r}_p = (x_1^p, x_2^p, \ldots, x_d^p)$. Just as in Ref. [4], the infinite summation in Eq. (7) can be truncated, through the well-known facts in the (inverse) Laplace transformation

$$\mathcal{L}_b\left[\Theta(\beta - k)\right] = e^{-kb}/b, \quad \mathcal{L}_\beta^{-1}\left[e^{-kb}/b\right] = \Theta(\beta - k),$$

with a positive constant k and the step function $\Theta(s)$ satisfying $\Theta(s) = 1$ for $s > 0$ and $\Theta(s) = 0$ for $s < 0$. The density matrix may thus be formally written as

$$\rho(\mathbf{r}', \mathbf{r}; t) = \mathcal{L}_\lambda^{-1}\left[\frac{1}{b}C(\mathbf{r}', \mathbf{r}; t; b)\right], \tag{11}$$

with $\lambda = M + (d+1)/2$. By explicitly carrying out the inverse Laplace transformation for the diagonal component of the density matrix, we find the particle density

$$\rho(\mathbf{r}, t) = \rho(\mathbf{r}, \mathbf{r}; t) = \frac{1}{l^d}\mathcal{D}(z), \tag{12}$$

where

$$z = (\mathbf{r} - \mathbf{r}_p)^2/l^2 \tag{13}$$

and the function $\mathcal{D}(z)$ is defined as [4]

$$\mathcal{D}(z) = \frac{e^{-z}}{\pi^{d/2}}\sum_{n=0}^{M}(-1)^n F_{M-n}^{(d)} L_n(z), \tag{14}$$

with $F_{2n}^{(d)} = (d/2 + 2n)\Gamma(d/2 + n)/[n!(d/2)!]$, $F_{2n+1}^{(d)} = 2\Gamma(d/2 + n + 1)/[n!(d/2)!]$ and $L_n(z)$ being the Laguerre polynomial.

The probability current density defined by

$$\mathbf{S}(\mathbf{r}, t) = \frac{\hbar}{2im}{\sum}' \left(\psi_\mathbf{n}^*(\mathbf{r}, t)\nabla\psi_\mathbf{n}(\mathbf{r}, t) - \psi_\mathbf{n}(\mathbf{r}, t)\nabla\psi_\mathbf{n}^*(\mathbf{r}, t)\right) \tag{15}$$

with $\sum'$ denoting the truncated summation of Eq. (6), can be found as

$$\mathbf{S}(\mathbf{r}, t) = \frac{\hbar}{im}\nabla_\mathbf{s}\rho(\mathbf{r}', \mathbf{r}; t)|_{\mathbf{s}\to 0} = \mathbf{v}\rho(\mathbf{r}, t), \tag{16}$$

where

$$\mathbf{v} = \dot{\mathbf{r}}_p + (\dot{l}/l)(\mathbf{r} - \mathbf{r}_p). \tag{17}$$

Up to the phase, the wave function $\psi_\mathbf{n}(\mathbf{r}, t)$ can be obtained from that of an eigenstate by isotropically rescaling the space according to the ratio l and globally displacing it by the amount $\mathbf{r}_p$. Since, under the rescaling, $\dot{l}/l$ plays the same role as the Hubble constant in the expanding universe, $\mathbf{v}$ is the velocity of the position $\mathbf{r}$ under the transformations, which explains the classical aspect of $\mathbf{S}(\mathbf{r}, t)$.

For the kinetic energy density, we explore the quantity

$$\tau_\mathbf{s}(\mathbf{r}, t)$$

$$= -\frac{\hbar^2}{8m}{\sum}'[(\nabla^2\psi^*_{\mathbf{n}}(\mathbf{r},t))\psi_{\mathbf{n}}(\mathbf{r},t) + \psi^*_{\mathbf{n}}(\mathbf{r},t)\nabla^2\psi_{\mathbf{n}}(\mathbf{r},t) \\ - 2(\nabla\psi^*_{\mathbf{n}}(\mathbf{r},t))\cdot\nabla\psi_{\mathbf{n}}(\mathbf{r},t)], \quad (18)$$

which can be evaluated as

$$\tau_{\mathbf{s}}(\mathbf{r},t) = -\frac{\hbar^2}{2m}\mathcal{L}^{-1}_{\lambda}\left[\frac{1}{b}\nabla^2_{\mathbf{s}}C(\mathbf{r}',\mathbf{r};t;b)\right]\Bigg|_{\mathbf{s}\to 0} \\ = \frac{m\mathbf{v}^2}{2}\rho(\mathbf{r},t) + \tau_Q(z,t), \quad (19)$$

where

$$\tau_Q(z,t) = \frac{d(2\pi)^{-d/2}\hbar^2}{4ml^{d+2}}\mathcal{L}^{-1}_{\lambda}\left[\frac{\coth(\frac{b}{2})\exp[-z\tanh(\frac{b}{2})]}{b\sinh^{d/2}(b)}\right]. \quad (20)$$

By carrying out the inverse Laplace transformation explicitly, we find that

$$\tau_Q(z,t) = \frac{\hbar^2 d e^{-z}}{4m\pi^{d/2}l^{d+2}}\sum_{n=0}^{M}(-1)^n G^{(d)}_{M-n}L_n(z), \quad (21)$$

with $G^{(d)}_{2n} = (32n^2 + d^2 + 16nd + 2d)\Gamma(d/2+n)/[4(n!)\Gamma(d/2+2)]$, and $G^{(d)}_{2n+1} = 2(4n+d+2)\Gamma(d/2+n+1)/[n!\Gamma(d/2+2)]$ [4]. $\tau_Q(z,t)$ can be represented in terms of $\rho(\mathbf{r},t)$ as

$$\tau_Q(z,t) = \\ \frac{d}{d+2}\frac{\hbar^2}{m}\left[\frac{1}{8}\nabla^2\rho(\mathbf{r},t) + \frac{\left(M+\frac{d+1}{2}-\frac{z}{2}\right)}{l^2}\rho(\mathbf{r},t)\right], \quad (22)$$

which, for the ground state, reduces to the known relation [4, 13]. An expression of the kinetic energy density is then written as

$$\begin{aligned}\tau(\mathbf{r},t) &= \frac{\hbar^2}{2m}{\sum}'[(\nabla\psi^*_{\mathbf{n}}(\mathbf{r},t))\cdot\nabla\psi_{\mathbf{n}}(\mathbf{r},t)] \\ &= \frac{m\mathbf{v}^2}{2}\rho(\mathbf{r},t) + 2\frac{d+1}{d}\tau_Q(z,t) \\ &\quad -\frac{\hbar^2}{ml^2}\left(M+\frac{d+1}{2}-\frac{z}{2}\right)\rho(\mathbf{r},t). \quad (23)\end{aligned}$$

$\rho(\mathbf{r}',\mathbf{r};t)$ satisfies the following Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\rho(\mathbf{r}',\mathbf{r};t) \\ = -\frac{\hbar^2}{2m}(\nabla^2 - \nabla'^2)\rho(\mathbf{r}',\mathbf{r};t) \\ + \left[\frac{mw^2(t)}{2}(\mathbf{r}^2 - \mathbf{r}'^2) - (\mathbf{r}-\mathbf{r}')\cdot\mathbf{F}(t)\right]\rho(\mathbf{r}',\mathbf{r};t). \quad (24)$$

which can be used to find conservation laws [14]. The coincidence limit $\mathbf{r} \to \mathbf{r}'$ of Eq. (24) gives

$$\frac{\partial}{\partial t}\rho(\mathbf{r},t) + \nabla \cdot [\rho(\mathbf{r},t)\mathbf{v}] = 0. \tag{25}$$

The coincidence limit after applying $\nabla_{\mathbf{s}}$ on Eq. (24) yields

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\rho(\mathbf{r},t)\mathbf{v}_i\right] + \frac{\partial}{\partial x_j}\left[\rho(\mathbf{r},t)\mathbf{v}_i\mathbf{v}_j\right] + \frac{4x_i}{mdl^2}\frac{\partial}{\partial z}\tau_Q(z,t)\\ &= -\rho(\mathbf{r},t)[w^2(t)x_i - \frac{F_i(t)}{m}].\end{aligned} \tag{26}$$

Similarly, by applying $\nabla_{\mathbf{s}}^2$ on Eq. (24) with the help of Eq. (11), we find that

$$\frac{\partial}{\partial t}\tau_Q(z,t) + \frac{\partial}{\partial x_i}\left[\tau_Q(z,t)\mathbf{v}_i\right] = -2\frac{\dot{l}}{l}\tau_Q(z,t). \tag{27}$$

For the ground state of $w(t) = w_c$ and $F_i(t) = 0$, Eq. (26) reduces to the known relation [15].

The momentum distribution defined by

$$n(\mathbf{p},t) = \frac{1}{(2\pi)^d}\int d\mathbf{r}d\mathbf{r}' e^{-i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')/\hbar}\rho(\mathbf{r}',\mathbf{r};t) \tag{28}$$

can be calculated as

$$\begin{aligned} n(\mathbf{p},t) &= \frac{\alpha^{-d}(t)}{(2\pi)^{d/2}}\mathcal{L}_\lambda^{-1}\left[\frac{e^{-\left[(\mathbf{p}-m\dot{\mathbf{r}}_p)^2\tanh(\frac{b}{2})/\alpha^2(t)\right]}}{b\sinh^{d/2}(b)}\right]\\ &= \alpha^{-d}(t)\mathcal{D}\left((\mathbf{p}-m\dot{\mathbf{r}}_p)^2/\alpha^2(t)\right),\end{aligned} \tag{29}$$

where

$$\alpha(t) = \sqrt{\hbar^2/l^2 + m^2\dot{l}^2}. \tag{30}$$

The momentum distribution in the momentum space thus has the identical shape of the particle density in coordinate space upon displacement and rescaling, as has been estimated for the one-dimensional free expanding gas [8]. Just as in the particle density (see, e.g., [1, 4, 16]), with the particle number N, the momentum distribution goes over to the approximate form

$$n_{TF} = \frac{\left[2(d!N)^{1/d} - \alpha^{-2}(t)(\mathbf{p}-m\dot{\mathbf{r}}_p)^2\right]^{d/2}}{2^d\pi^{d/2}\Gamma(1+d/2)\alpha^d(t)}, \tag{31}$$

for $2(d!N)^{1/d} > (\mathbf{p}-m\dot{\mathbf{r}}_p)^2/\alpha^2(t)$, in the large N limit, and the exact profile has corrugation-like corrections from the approximation [1, 4]. For a time-constant trap, if we choose $u =$

$\cos(w_c t)$, $v = \sin(w_c t)$, and $x_i^p = 0$, n_{TF} exactly agrees with the distribution found for the deep BCS limit of the atomic Fermi gas through the mean-field theory [3] (see also Ref. [2]).

For a general $w(t)$, l satisfies $m\ddot{l} + mw^2(t)l - \hbar^2/(ml^3) = 0$, and the behavior of l has been analyzed for time-periodic $w(t)$ [11] to find that l could be more and more amplified in general and become unstable as time passes. For a time-constant trap, as in the case after the modulation of the frequency is stopped, l satisfies $m^2\dot{l}^2 + \hbar^2/l^2 = -m^2 w_c^2 l^2 + 2m\varepsilon = \alpha^2(t)$ with a constant ε, which shows, *as a manifestation of the uncertainty principle* [17], $\alpha(t)$, characterizing the dispersion of the momentum distribution, increases (decreases) when l or the dispersion of the particle density is decreased (increased). For the time-constant trap, up to a global time-displacement, l is written without losing generality as $l = \sqrt{\hbar/(mw_c)}\sqrt{A^2 \cos^2 w_c t + \sin^2 w_c t/A^2}$, with a positive constant A [6]. In this case, $\alpha(t) = \sqrt{m\hbar w_c[A^2 \sin^2 w_c t + \cos^2 w_c t/A^2]}$, which shows, the dispersion of the momentum distribution could be *arbitrarily large or small* in the limit of $A \to 0$ or $A \to \infty$.

The formalisms developed are also valid for $w(t) = 0$ in d-dimension, with $u = a_1$, $v = a_2(t - t_0) + a_3$ of a_1, a_2, a_3, t_0 being constants. For a sudden and total opening of the trap at $t = t_0$, the continuity of the wave functions requires the continuities of x_i^p, $\dot{x}_i^p$, η, $\dot{\eta}$, which gives $a_2 = \sqrt{\Omega^2/(m\eta_-)^2 + \dot{\eta}_-^2}$, $a_1 = \frac{\Omega}{ma_2}$, $a_3 = \frac{\eta_-\dot{\eta}_-}{a_2}$, where the subscript $-$ denotes that the quantity is evaluated in the limit of $t - t_0 \to -0$. If $\dot{\eta}_- < 0$, the dispersion of the particle density, which is proportional to l, will decrease after the opening of the trap until $t - t_0 = |a_3|/a_2$, and then it will increase, while the cloud of the gas will continuously expand for $\dot{\eta}_- \geq 0$. $\alpha(t)$ of Eq. (30) with $l = \sqrt{\hbar[a_1^2 + (a_2(t-t_0) + a_3)^2]/(ma_1 a_2)}$ is a constant α_-, that is, for $t > t_0$, $\alpha(t) = \sqrt{m\hbar a_2/a_1} = \sqrt{\hbar^2/l_-^2 + m^2\dot{l}_-^2} = \alpha_-$. After the opening, *the momentum distribution of the gas is thus stationary*, as has been predicted for the one-dimensional expanding gas. If the opening is for the ground state of the trap of a constant frequency w_c, $l_- = \sqrt{\hbar/(mw_c)}$, and $\dot{l}_- = 0$, and thus $\alpha(t) = \sqrt{\hbar m w_c}$ [8].

In summary, the expressions of the particle, kinetic energy, probability current densities, and the momentum distribution have been derived for a coherent state of a noninteracting Fermi gas in a d-dimensional isotropic harmonic trap, and, for the relations of the densities, conservation laws have been given. It has been shown that the profile of the momentum distribution is identical in shape with that of the particle density, while the dispersion of the distribution increases (decreases) when that of the particle density is decreased (increased) in a time-constant trap. The dispersion of the distribution could be arbitrarily large or small,

which basically comes from the invariance under the STT and the uncertainty principle. Since the invariance exists only for restricted cases [6, 7], an appearance of the distribution of the large or small dispersion in a system of Fermi atoms may indicate that the system is in the noninteracting regime. After a total opening of the trap, the momentum distribution does not change from that of the gas at the moment of the opening, while, long after the opening, the dispersion of the particle density increases according to $\alpha_- t/m$ with the constant α_- which characterizes the dispersion of the distribution. As the uncertainties are *controlled by the classical solutions* in the coherent states, an experimental realization of the system described here may be important to provide an *observable* manifestation of the uncertainty principle.

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[16] P. Vignolo, A. Minguzzi, and M.P. Tosi, Phys. Rev. Lett. **85**, 2850 (2000).

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